Let $(f_n : I \to \mathbb{R})_{n\in\mathbb{N}}$ be a sequence of functions. If $f : I \to \mathbb{R}$ is another function, we say that f_n converges uniformly to f is for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for every $x \in \mathbb{R}$ we have $|f_n(x) - f(x)| < \epsilon$.

We say that f_n converges pointwise to f is for every $x \in \mathbb{R}$, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$.

Problem 1

1. Provide a sequence of functions $f_n : I \to \mathbb{R}$ that converges pointwise to f, yet f is not continuous.
2. Provide a sequence of integrable functions $f_n : I \to \mathbb{R}$ that converges pointwise to an integrable function f, yet the sequence $(\int f_n)$ does not converge to $\int f$.
3. Provide a sequence of integrable functions $f_n : I \to \mathbb{R}$ that converge pointwise to a non-integrable function f.
4. Provide a sequence of nonnegative bounded functions $f_n : I \to \mathbb{R}$, with $M_n = \sup f_n$, such that $\sum_{n=1}^{\infty} f_n$ converges uniformly, yet $\sum_{n=1}^{\infty} M_n$ does not converge. This shows the converge to the Weierstrass M-test does not hold.

Problem 2

Suppose that f_n are continuous functions on $[0, 1]$ that converge uniformly to f. Prove that

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n = \int_0^1 f.$$

Is this true if the convergence isn't uniform?

Problem 3

Prove that for $-1 < x \leq 1$,

$$\log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

Problem 4

1. Write down the power series for $\log(1 - x)$ and $\log[(1 + x)/(1 - x)]$ around $x = 0$.
2. Show that the power series for $f(x) = \log(1 - x)$ converges only for $-1 \leq x < 1$, and that the power series for $g(x) = \log[(1 + x)/(1 - x)]$ converges only for x in $(-1, 1)$.